Probability of a Coin Landing on Its Edge

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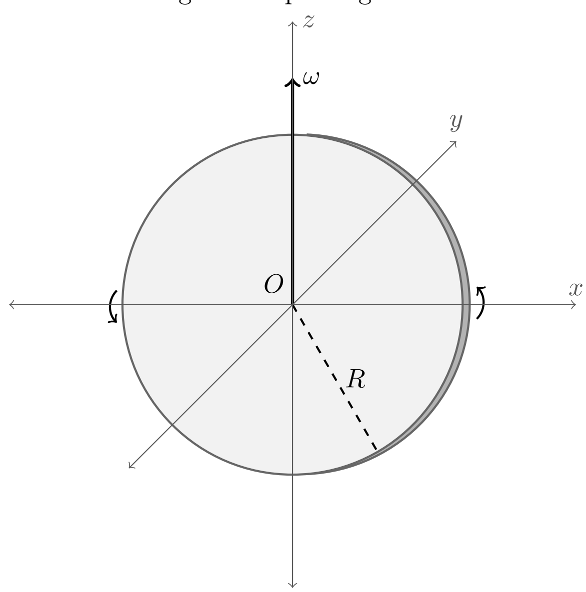
1 Introduction

The flipping of a coin is perhaps one of the most longstanding symbols of probability and chance that mathematicians have used. Of course, classically, we have the argument that by symmetry, the probability of a coin landing on heads or tails is *equal* - and this is true. However, the statement that it is $\frac{1}{2}$ for both sides is untrue! We have of course, neglected to consider the possibility of the coin landing on its side. "This is impossible!" one may exclaim. I thought the same. However, as I have begun learning physics, I have learned to explore the extremes of every system. Equilibrium in physics is often a delicate state, and if one does not have a thorough understanding of the system in question, they may only be able to discern the 'common sense' positions of equilibrium. When we roll a dice, it seems clear that it will land on one of its six faces; but why can it not land on a corner? After all, its center of mass can be directly above its point of contact with the surface, and in this case, it will be stable. Going further, I wondered if we flip an atomically sharp pencil a large number of times, if it could land on its tip? It turns out – due to a curious phenomenon caused by quantum mechanics, it *cannot*!

This exploration of the extremes has led me to wonder whether a flipping coin can achieve a stable equilibrium position on its **edge**.

Figure 1: Spinning Coin[1]



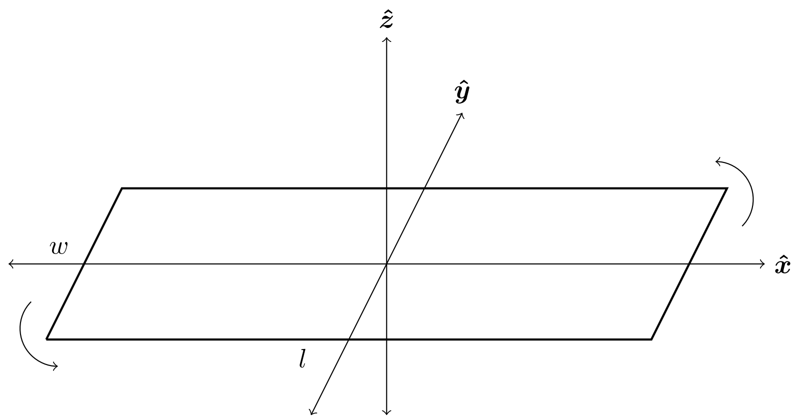
To explore this question, I decided to first model the coin with some idealizations and approximations. I hope to have sufficiently justified these approximations throughout this exploration, and in doing so, demonstrate that the overall model is realistic. This model has several equations that cannot be solved analytically for an exact solution, and so after developing this model, I implemented it into a computer simulation. Simply brute forcing over numbers in an intelligent fashion brings forth numeric solutions! The simulation proceeds to use these numeric solutions to iterate over coin tosses with different initial parameters. I then used the results of the simulation to calculate the probability of a coin landing on its edge.

[1] As a sign convention, angular velocity ω is taken to be positive when the coin is spinning in the counterclockwise direction (as shown in Figure 1).

2 Theory

There is an important quantity in rotational mechanics known as moment of inertia I. It acts as the rotational analogue of the linear mechanical quantity **mass**, and is thus extremely important in our discussion of the coin's rotation. To find an object's moment of inertia about a certain axis of rotation is quite simple - we integrate over every point in the object, using the fact that an infinitesimal point's moment of inertia about an axis is MR^2. For our cylindrical coin, we take the axis of rotation to be the $\hat{z}$ vector, known as the 'central diameter' of a cylinder. To begin finding a solid cylinder's moment of inertia about $\hat{z}$, we decompose it into a rectangles with infinitesimal thickness $d\alpha$, all rotating about their geometrical center. We then integrate over these rectangles to find the overall moment of inertia I.

Figure 2: Rectangle Spinning about $\hat{z}$



We begin by finding the moment of inertia of a rectangle about its geometrical center. We use the perpendicular axis theorem, stating that if an object is a plane figure, the moment of inertia about an axis perpendicular to the plane is equal to the sum of the moments of inertia about any two mutually perpendicular axes lying in the plane and intersecting at the perpendicular axis. It is known that the moment of inertia of the rectangle I_x through $\hat{x}$ is $\frac{mw^2}{12}$ and likewise, the moment of inertia I_y through $\hat{y}$ is $\frac{ml^2}{12}$. By the perpendicular axis theorem, $I_z = I_x + I_y = m\frac{w^2+l^2}{12}$.

We move on to apply this to our cylinder. By taking slices of thickness $d\alpha$, we see that each slice is a rectangle with a width equal to the thickness T of the cylinder. The length is variable. Now, we integrate over α, taking α to be the vertical distance of each rectangle from the center of the cylinder. Note that the length l is thus $2\sqrt{R^2-\alpha^2}$.

$$I = \int_{-R}^{R} \frac{T^2 + 4(R^2-\alpha^2)}{12} 2\rho T\sqrt{R^2-\alpha^2}\, d\alpha$$

The rectangle has a mass given by the area of the rectangle multiplied by its infinitesimal thickness $d\alpha$, multiplied by the density of the cylinder.

$$= \frac{\rho T}{6}\int_{-R}^{R} \Big(T^2 + 4(R^2-\alpha^2)\Big)\sqrt{R^2-\alpha^2}\, d\alpha$$
$$= \frac{\rho T}{6}\left(T^2\int_{-R}^{R}\sqrt{R^2-\alpha^2}\, d\alpha + 4\int_{-R}^{R}(R^2-\alpha^2)^{\frac{3}{2}}\, d\alpha\right)$$

The equation $\sqrt{R^2-\alpha^2}$ from $-R$ to R is the equation of a semicircle of radius R. Thus the area under this curve is $\boldsymbol{\pi R^2}$. The integral of $(R^2-a^2)^{\frac{3}{2}}$ is beyond the scope of this discussion[2]; the result is $\frac{3\pi R^4}{8}$.

$$
\begin{aligned}
&= \frac{\rho T}{6}\Big(\frac{\pi T^2 R^2}{2} + \frac{12\pi R^4}{8}\Big) \\
&= \frac{1}{6}\pi R^2 T\rho\Big(\frac{T^2}{2} + \frac{3R^2}{2}\Big) \\
&= M\Big(\frac{T^2}{12} + \frac{R^2}{4}\Big) \\
&= \frac{MT^2 + 3MR^2}{12}
\end{aligned}
$$

In a coin, thickness T is very small compared to its radius R. If the two are in the ratio $T : R = \beta$ where $\beta \ll 1$, we can see that $\frac{T^2}{12} + \frac{R^2}{4} = \frac{\beta^2 R^2}{12} + \frac{R^2}{4}$. As β approaches smaller values, the $\frac{\beta R^2}{12}$ term falls off rapidly, so that we can approximate the expression to be $\frac{R^2}{4}$. Thus we approximate the moment of inertia of the cylinder to be $\frac{MR^2}{4}$.

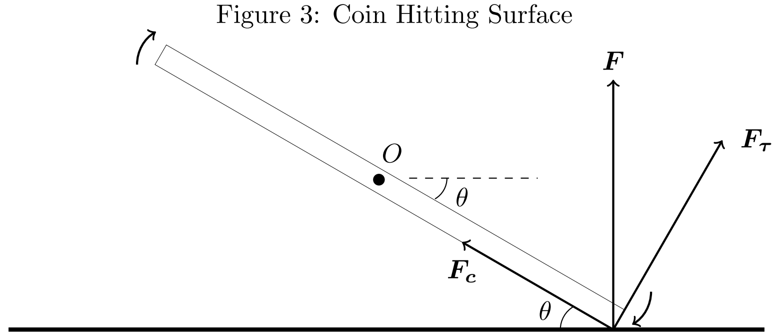
3 Model

We wish to model the coin's interactions with the surface that it's bouncing on. To start, we must make a number of assumptions:

- The coin is a perfect cylinder with radius R and thickness T, and has uniform density ρ. Thus the mass M of the coin is $\rho\pi R^2 T$.
- The coin's rotation is always entirely about the z axis, as shown in Figure 1.
- The impulse J delivered by the surface on the coin on each successive bounce is related to the normal velocity v_N of the point that hits the table by a proportionality constant γ, such that $J = \gamma|v_N|$. The duration of the contact between coin and surface is negligibly short such that the effect of other forces on the coin (i.e. gravity) are negligible.
- Air resistance is negligible and the acceleration due to gravity is g.
- The coin is flipped such that it has an initial angular momentum of L and its center of mass O has a velocity v_{cm}. It is flipped at the same level as the surface (initial height $h = 0$)

This model equips us with enough to begin analyzing what happens when the coin collides with the surface.

Figure 3: Coin Hitting Surface



[2]It is possible by using a substitution $\alpha = R\sin u$, thus leaving the integral in the form $R^4\int\cos^4(u)du$. This can be solved using the reduction formula $\int\cos^n u\ dx = \frac{n-1}{n}\int\cos^{n-2}u\ du + \frac{\cos^{n-1}u\sin u}{n}$.

We study the cross section of the coin when it hits the surface at a certain angle θ. There are three forces acting on the coin during the collision: one force due to the electrostatic repulsion between the coin and the table (labelled F), the gravitational force F_g on the coin, and the friction force F_{fr}. Since the duration of the collision is short (on the order of 10^{-2} seconds), the impact of this force is negligible. The horizontal friction force also acts on the coin due to the coin's horizontal velocity at the time of impact, providing an additional torque. However, again, the static friction coefficient μ is not high enough to provide a significant torque over the tiny duration of the impact. On the other hand, the normal force F is significant. We separate F into two components F_c, which does not rotate the coin, and F_τ, which purely rotates the coin.

$$F_c \cos\theta - F_\tau \sin\theta = 0 \tag{a}$$
$$F_c \sin\theta + F_\tau \cos\theta = F \tag{b}$$

By (a), we have $F_c = F_\tau \frac{\sin\theta}{\cos\theta} = F_\tau \tan\theta$. Substituting this into (b), we get:[3]

$$\begin{aligned}
F_\tau \tan\theta \sin\theta + F_\tau \cos\theta &= F \\
F_\tau \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta} \right) &= F \\
F_\tau \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} &= F \\
F_\tau &= F\cos\theta \qquad (1) \\
F_c &= F\sin\theta \qquad (2)
\end{aligned}$$

However, we must note that we have no idea what F might be - interestingly, we also have no need to solve for F. Simply using the relation for impulse $J = \gamma|v_N|$ allows us to find the information we need, since $J = \int F\ dt$. Our goal is to find the change in angular velocity $\Delta\omega$ and the change in center of mass velocity Δv_{cm} due to this collision. We begin by finding Δv_{cm}:

$$\Delta v_{cm} = \int a\ dt$$

We are only interested in the change in vertical velocity of the coin - so we take only the vertical component of F_c.

$$\begin{aligned}
&= \frac{\int F_c \sin\theta\ dt}{M} \\
&= \frac{\int F \sin^2\theta\ dt}{M} \\
&= \frac{\sin^2\theta}{M} \int F\ dt \\
&= \frac{\sin^2\theta}{M} J \\
&= \frac{\gamma|v_N|\sin^2\theta}{M}
\end{aligned}$$

Finding the change in angular velocity $\Delta\omega$ is slightly more complicated. The torque delivered by the force F_τ is related to torque by $\tau = R \times F_\tau$. Since F_τ is at right angles to R, we can see that $|\tau| = RF_\tau$. Torque is the rate of change of angular momentum $\frac{dL}{dt}$, so we see that the change in angular momentum $|\Delta L| = \int RF_\tau\ dt$. Thus, by a similar process to our last calculations, we see that $|\Delta L| = \gamma R|v_N|\cos\theta$. However, L is related to ω by $L = I\omega$, and thus:

$$\begin{aligned}
|\Delta\omega| &= \frac{\gamma R|v_N|\cos\theta}{I} \\
&= \frac{4\gamma|v_N|\cos\theta}{MR}
\end{aligned}$$

[3] Of course, our result (1) and (2) is clear when considering F_τ and F_c as the edges of a rectangle, and taking F to be the diagonal.

The direction of $\Delta\omega$ (whether it is positive or negative) depends on the angle θ at which the coin hits the surface. By inspecting the two cases where $\theta < \frac{\pi}{2}$ and $\theta > \frac{\pi}{2}$, we see that **when $\theta < \frac{\pi}{2}$, ω decreases**, and vice versa. Thus we have $\Delta\omega = -\frac{4\gamma|v_N|\cos\theta}{MR}$.

It is now appropriate to find out what v_N is. It is quite simple - in fact, it is the vector sum of the edge of the coin's rotational velocity (normal to the surface) and the coin's center of mass velocity:

$$v_N = -\omega R\cos\theta + v_{cm}$$

We have a term $-\omega R\cos\theta$ because the edge of the coin is moving **downwards** at $\theta < \frac{\pi}{2}$ and vice versa. Since the $|v_N|$ we have been using is assumed to be positive in all our calculations and v_N is negative, we can now say that $|v_N| = \omega R\cos\theta - v_{cm}$. Plugging this in to our equations, we see:

$$\Delta v_{cm} = \frac{\gamma(\omega R\cos\theta - v_{cm})\sin^2\theta}{M} \tag{3}$$

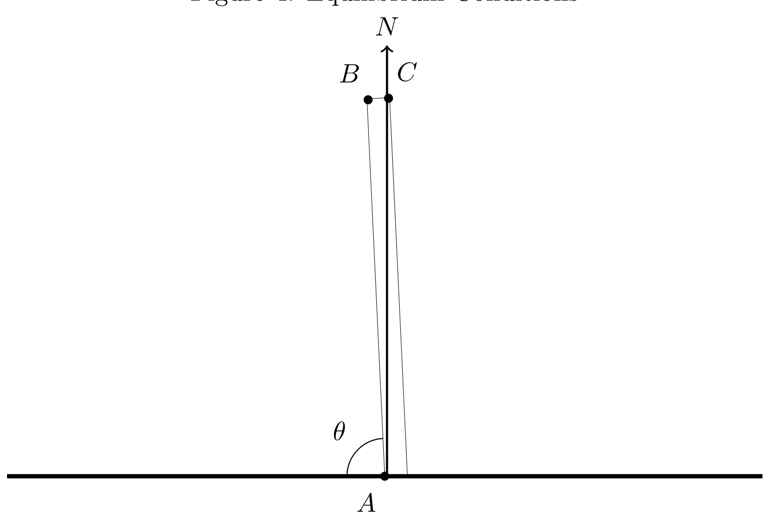
$$\Delta\omega = \frac{4\gamma(v_{cm} - \omega R\cos\theta)\cos\theta}{MR} \tag{4}$$

The two equations (3) and (4) are the key to analyzing the coin's motion.

3.1 Final State

The coin will be considered to have settled, or be in a position to settle on a particular side in a specific set of conditions. We call this state of settling or preparedness to settle 'equilibrium'. Since we are only interested in the case where the coin reaches equilibrium on its edge, we solve only for this specific case. If the coin had no center of mass velocity and no angular velocity, the following would be considered a case in which it was about to settle on its side:

Figure 4: Equilibrium Conditions



In the case where $v_{cm} = 0$ and $\omega = 0$, this is considered to be a coin in equilibrium. The right triangle $\triangle ABC$ is the key to equilibrium, where A is the point of contact of the coin with the surface, and C is the opposite corned to A. C must be directly above A, as shown by the vertical vector N that runs through A and C. Since the angle θ characterizes equilibrium, we wish to find it. By simply geometry we see that $\angle BAC = \arctan\frac{T}{2R}$. Since $\theta + \angle BAC = \frac{\pi}{2}$, $\theta = \frac{\pi}{2} - \arctan\frac{T}{2R}$. By symmetry, another acceptable θ is $\frac{\pi}{2} + \arctan\frac{T}{2R}$. For brevity, we label $\arctan\frac{T}{2R}$ as ϵ. Thus $\theta = \frac{\pi}{2} \pm \epsilon$. So our conditions for the special equilibrium is:

$$\frac{\pi}{2} - \epsilon \le \theta \le \frac{\pi}{2} + \epsilon \tag{5}$$

Of course, $v_{cm} \neq 0$ (it is less than 0 at time of impact) and $\omega \neq 0$. Instead, we use (3) and (4) to predict the new velocity v'_{cm} and angular velocity ω' of the coin after the collision. Since $\theta \approx \frac{\pi}{2}$, we can approximate $\sin\theta \approx 1$ and $\cos\theta \approx 0$ and use a much more simplified form of (3) and (4):

$$v'_{cm} \approx v_{cm} - \frac{\gamma v_{cm}}{M} = v_{cm}\Big(1 - \frac{\gamma}{M}\Big)$$
$$\omega' \approx \omega + \frac{4\gamma v_{cm} \cdot 0}{MR} = \omega$$

The coin will end up in another equilibrium with (5) if and only if a certain condition is met, which we now investigate. If the coin is to land again where $\theta \approx \frac{\pi}{2}$, we can safely estimate that the time t that the coin spends in the air is $\frac{2v'_{cm}}{g}$ by simple kinematics. Now, since we wish for the coin to end up in a new equilibrium with (5) after the collision, we must restrict the angle Θ rotated through by the coin to certain values. We see that in the first collision, there exists a 'padding' between θ and its two boundaries. θ exceeds the minimum allowed value by a certain amount $C_a = \theta - \frac{\pi}{2} + \epsilon$ and is also under the maximum allowed value by an amount $C_b = \frac{\pi}{2} + \epsilon - \theta$. So, the coin may turn through any multiple of π, and then an **additional** angle C_b - or it may turn through a small angle C_a **less** any multiple of π. Formally:

$$n\pi - C_a \leq \Theta \leq n\pi + C_b$$

We know the angle turned through by the coin is $\Theta = \omega' t \approx \omega\frac{2v'_{cm}}{g}$. Thus:

$$n\pi - C_a \leq \frac{2\omega v'_{cm}}{g} \leq n\pi + C_b$$
$$\frac{g(n\pi - C_a)}{2} \leq v'_{cm}\omega \leq \frac{g(n\pi + C_b)}{2} \tag{6}$$

However, we may further reduce (6) by observing something very simple. Since angular velocity remains constant, rotational kinetic energy remains constant. Therefore, kinetic energy must be strictly decreasing since energy losses due to friction and air resistance are inevitable - and along with this, a strictly decreasing velocity v_{cm}. This means that a 'final' equilibrium position cannot be reached unless $n = 0$. If $n \geq 1$, the velocity v'_{cm} will begin to decrease, and since our condition (6) is very sensitive[4] to changes in v_{cm}. As v_{cm} is continually decreasing in small increments, it will inevitably break (6). Thus, the only case where we can reach a *final* equilibrium of sorts is where $n = 0$ and v'_{cm} is small[5]. Thus:

$$-\frac{gC_a}{2} \leq v'_{cm}\omega \leq \frac{gC_b}{2}$$
$$-\frac{g(\theta - \frac{\pi}{2} + \epsilon)}{2} \leq v_{cm}\omega\frac{M-\gamma}{M} \leq \frac{g(\frac{\pi}{2} + \epsilon - \theta)}{2}$$

Note that $\frac{M-\gamma}{M} < 0$.

$$\frac{gM(\frac{\pi}{2} - \theta + \epsilon)}{2(M-\gamma)} \leq v_{cm}\omega \leq \frac{gM(\frac{\pi}{2} - \theta - \epsilon)}{2(M-\gamma)} \tag{7}$$

3.2 Restricting γ

Up until now, we have been operating under the assumption that γ can simply take any positive value. However, this is not so - we can see why by taking the case where $\theta = \frac{\pi}{2}$ and $\omega = 0$. We have already calculated that the new velocity v'_{cm} after collision in this case is related to the old v_{cm} by: $v'_{cm} = v_{cm}(1 - \frac{\gamma}{M})$. In the case where $\theta = \frac{\pi}{2}$ and $\omega = 0$, the coin has no rotational kinetic energy and acts as a normal bounding object. Thus, we can introduce a physical quantity called the *coefficient of restitution*. This coefficient k is

[4] We see that v'_{cm} can only take a range of values of size $\frac{g}{2\omega}(C_a + C_b) = \frac{g\epsilon}{\omega}$. Since $\epsilon \approx 0$, we see that any small changes in v_{cm} will immediately break (6).

[5] This is because if is too high, the coin will have the chance to turn again after the next bounce that we have predicted for. By setting a minimum bound on the next velocity v'_{cm}, we can consider the next bounce to be the *final* bounce. It turns out that changing the sensitivity of the small v'_{cm} condition has virtually no effect after we set the condition to $v'_{cm} \leq 0.01$.

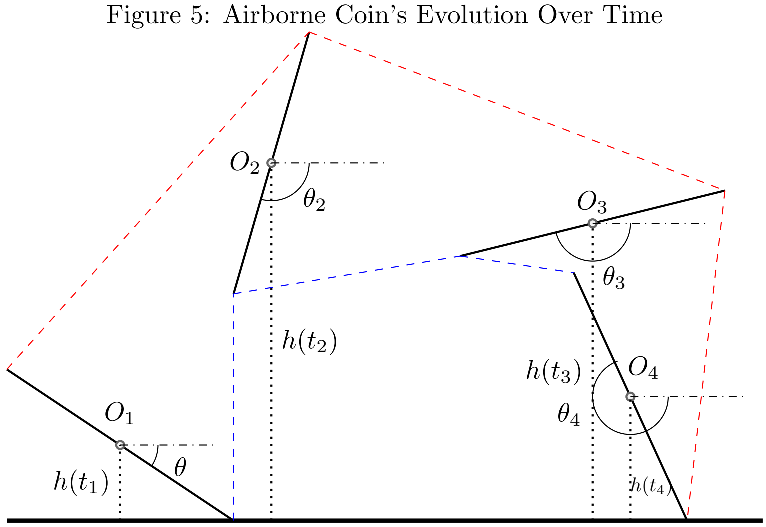
the ratio of the initial to the final speed of an object after collision: $e = \frac{|v'_{cm}|}{|v_{cm}|}$. By conservation of energy, of course, this quantity $e < 1$. It is characterized by the two objects involved in the collision. This is the **true**[6] physical quantity that we must use, not γ, so we wish to express γ in terms of e:

$$\begin{aligned} v'_{cm} &= v_{cm}\left(1 - \frac{\gamma}{M}\right) \\ |v'_{cm}| &= |v_{cm}|\left(\frac{\gamma}{M} - 1\right) \\ \frac{1}{\frac{\gamma}{M} - 1} &= \frac{|v_{cm}|}{|v'_{cm}|} \\ \frac{M}{\gamma - M} &= \frac{1}{e} \\ \gamma &= M(e+1) \end{aligned} \tag{8}$$

3.3 Finding θ

The final variable that we have yet to solve for is the variable θ of collision. Of course, the current collision angle θ depends on the last collision angle $\bar{\theta}$, along with the current angular velocity ω and center of mass velocity v_{cm}. By approximating the coin as a line, we can find an expression for the height z of any point in the coin at a certain radius R. We note that the relationship between the last angle $\bar{\theta}$ and the current angle θ *is the same* is the relationship between the current angle θ and the next angle θ'. Thus, we study how θ and θ' are related.

Figure 5: Airborne Coin's Evolution Over Time



The height h of the center of mass O over time is shown in the diagram. The path of the ends of the coin are shown in red and blue. With this diagram, it becomes easy to find the height z of any point on the coin at a radius r. We can see that $h(t) = h(t_1) + v'_{cm}t - \frac{gt^2}{2}$. However, by simple geometry we can see that $h(t_1) = R\sin\theta$, so that

$$h(t) = R\sin\theta + v'_{cm}t - \frac{gt^2}{2}$$

The angle $\Theta(t)$ evolves over time with the angular velocity and the initial angle θ so that $\Theta(t) = \theta + \omega' t$. The height z of any point on the coin at a radius[7] r away from the center O is thus the vector sum of the

[6] γ is, in fact, dependent on the mass M of the object colliding with the surface, as shown in (8). Thus, it cannot be said to be a true physical quantity, only a mathematical construct for the sake of brevity.

[7] We take the end of the coin with the blue path to be $+R$ and the end with the red path to be $-R$.

height of the center of the coin $h(t)$ and vertical distance from that point to the center:

$$\begin{aligned} z(r,t) &= h(t) - r\sin\big(\Theta(t)\big) \\ &= R\sin\theta + v'_{cm}t - \frac{gt^2}{2} - r\sin(\theta + \omega' t) \end{aligned}$$

We wish to solve $z(r,t) = 0$, where $t \neq 0$ so that we may find at what angle the coin hits the surface again. However, we note that it is only possible for the coin to hit the surface where $r = \pm R$, so we study only $z(R,t)$ and $z(-R,t)$:

$$z(\pm R, t) = R\sin\theta \mp R\sin(\theta + \omega' t) + v'_{cm}t - \frac{gt^2}{2}$$

This equation can be solved numerically by trying values for t in small increments and taking the value where $z(\pm R, t) = 0$ first. We call this solution t_0. Despite the fact that this solution can only be obtained numerically, it is interesting to approximate a value for t by observing that the height $h(t_1) \approx h(t_4)$, at most differing by R. So if $0 \leq h(t_4) \leq h(t_1) + R$, we can see that:

$$\begin{aligned} &0 \leq h(t_1) + v'_{cm}t_0 - \frac{gt_0^2}{2} \leq h(t_1) + R \\ &0 \leq h(t_1) + v'_{cm}t_0 - \frac{g}{2}t_0^2 \textbf{ and } v'_{cm}t_0 - \frac{g}{2}t_0^2 \leq R \\ &0 \geq \frac{g}{2}t_0^2 - v'_{cm}t_0 - h(t_1) \textbf{ and } 0 \leq \frac{g}{2}t_0^2 - v'_{cm}t_0 + R \\ &\frac{v'_{cm} - \sqrt{(v'_{cm})^2 + 2gh(t_1)}}{g} \leq t_0 \leq \frac{v'_{cm} + \sqrt{(v'_{cm})^2 + 2gh(t_1)}}{g} \end{aligned}$$

$$\textbf{and}$$

$$t_0 \leq \frac{v'_{cm} - \sqrt{(v'_{cm})^2 - 2gR}}{g} \textbf{ or } t_0 \geq \frac{v'_{cm} + \sqrt{(v'_{cm})^2 - 2gR}}{g}$$

We simplify these two conditions to:

$$\frac{v'_{cm} - \sqrt{(v'_{cm})^2 + 2gR\sin\theta}}{g} \leq t_0 \leq \frac{v'_{cm} - \sqrt{(v'_{cm})^2 - 2gR}}{g} \tag{9}$$

$$\textbf{or}$$

$$\frac{v'_{cm} + \sqrt{(v'_{cm})^2 - 2gR}}{g} \leq t_0 \leq \frac{v_{cm} + \sqrt{v_{cm}^2 + 2gR\sin\theta}}{g} \tag{10}$$

We try values of t in small increments according to (9) and (10) to solve numerically for $z(\pm R, t) = 0$.

Using equations (9) and (10), we may numerically solve for the duration t_0 of a coin's time spent in the air between collisions - this gives us the coin's next collision angle θ' in terms of the coin's current collision angle θ by the equation[8]:

$$\theta' = (\theta + \omega' t_0) \ \% \ \pi \tag{11}$$

We go further by predicting the coin's *next velocity of collision*, which we call $\bar{v}_{cm}$, given its velocity after bouncing back from the current collision v'_{cm}:

$$\bar{v}_{cm} = v'_{cm} - gt_0 \tag{12}$$

[8]We use the % symbol to imply remainder, instead of writing congruence and mod.

3.4 Equations of Interest

To review the terminology and symbols used to this point, we have:

Table 1: Symbols and Notations Used

Symbol	Meaning	Sign Convention
v_{cm}	Velocity of the center of mass of the coin upon impact with the surface.	Negative
v'_{cm}	Velocity of the center of mass of the coin immediately after impact with the surface.	Positive
ω	Angular velocity of the coin upon impact with the surface.	Positive for counterclockwise rotation and negative for clockwise.
ω'	Angular velocity of the coin immediately after impact with the surface.	The same as ω.
θ	Angle at which coin hits the surface	Positive
θ'	Angle at which coin hits the surface *next*	Positive
t_0	Duration that the coin spends in the air between consecutive collisions.	Positive
$\bar{v}_{cm}$	Velocity of the coin on the *next* collision with the surface (velocity of the coin after being accelerated by g for t_0 seconds).	Negative
e	Coefficient of restitution.	$0 \leq e \leq 1$
R, T	Radius and thickness of the coin, respectively.	Positive
g	Magnitude of acceleration due to gravity.	Positive

With this, we are now in a position to rewrite our three governing equations (3), (4), (7) in a more convenient form using a simplified γ. The equations of motion:

$$v'_{cm} = v_{cm} + (e+1)(\omega R\cos\theta - v_{cm})\sin^2\theta \tag{13}$$

$$\omega' = \omega + \frac{4(e+1)(v_{cm} - \omega R\cos\theta)\cos\theta}{R} \tag{14}$$

And the equilibrium condition:

$$\frac{-g(\frac{\pi}{2} - \theta + \epsilon)}{2e} \leq v_{cm}\omega \leq \frac{-g(\frac{\pi}{2} - \theta - \epsilon)}{2e} \textbf{ and } v'_{cm} \leq 0.01$$

$$\frac{g}{2e}\left(\theta - \frac{\pi}{2} - \arctan\frac{T}{2R}\right) \leq \omega v_{cm} \leq \frac{g}{2e}\left(\theta - \frac{\pi}{2} + \arctan\frac{T}{2R}\right) \textbf{ and } v'_{cm} \leq 0.01 \tag{15}$$

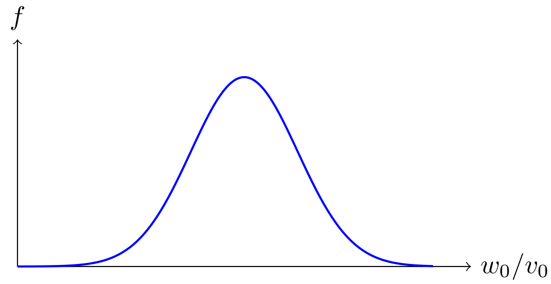
4 Simulation

The equations (11), (12), (13), (14) and (15) gives us the necessary tools to create a simulation that numerically solves these systems over each successive bounce. Thus we proceed by iterating over each bounce, calculating the coin's next velocity via (13) and its next angular velocity via (14). We end iteration when either our equilibrium condition (15) is met, or $v'_{cm} \leq 0.01$. When v'_{cm} gets so small that $v'_{cm} \leq 0.01$, it is clear that no further turning progress can be made by the coin, since v'_{cm} is strictly decreasing as seen by (13). The problem now is strictly in implementation. The implementation for this model is available here (https://github.com/andigu/Coin-Simulation).

4.1 Initial Conditions

We note that the coin's initial angular velocity ω_0 and initial center of mass velocity v_0 are by no means fixed with a human 'flipper'. We assume that these two variables take the form of a bell curve, so that our code will give the coin an initial ω_0 and initial v_0 by random selection across a bell curve.

Figure 6: Frequency f of an Initial ω_0 or v_0

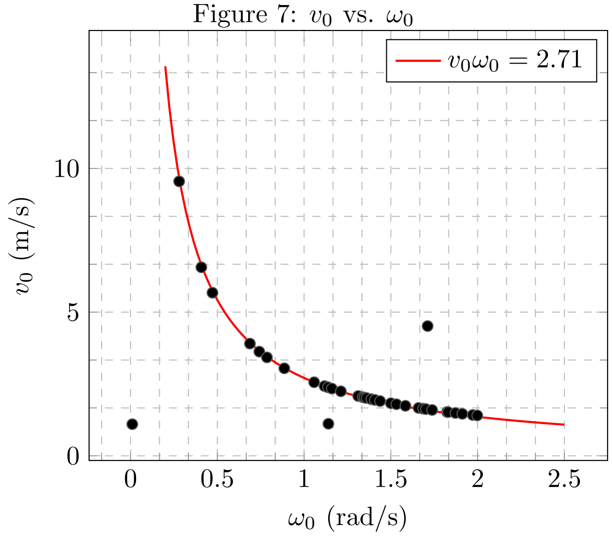


4.2 Results

The simulation was run over ω_0 from 0 through 2 in increments of 0.002 for 1000 iterations, and over all v_0 from 1 through 11 in increments of 0.005 for 2000 iterations. Running over each combination of ω_0 and v_0 yielded a total of $2 \cdot 10^6$ iterations. The remainder of parameters remained fixed, with the dimensions of the coin closely matching a Canadian quarter, $g = 9.8\frac{m}{s^2}$, and $e = 0.5$.

The number of times and initial configuration ω_0/v_0 of the coin when the coin landed on its edge were recorded. Results indicated that there were 35 pairs ω_0/v_0 that were successful from 2 million trials. This gives a naive probability of $\frac{35}{2 \cdot 10^6} = 0.00175\%$. Putting these pairs onto a scatter plot, a strong relationship immediately emerges.

Figure 7: v_0 vs. ω_0



This trendline $v_0\omega_0 = 2.71$ is an especially strong fit, with a coefficient of determination of 0.8968. Yet furthermore, this gives us a strong insight into the nature of this problem, especially in reference to (15). We see that this product $v\omega$ is a natural constraint in the problem - and of course emerges as a restriction in the initial condition of the coin.

4.3 Weighted Probability

Of course, the naive probability of $\sim 0.00175\%$ is not entirely accurate. As illustrated in Figure 6, it is not equally likely that any given ω_0 or v_0 is chosen. So instead we model both these variables along a bell curve and observe the domain 0 through 2 for ω_0 and the domain 1 through 11 for v_0. We center the bell curve for each at the midpoint of the domain and give it a standard deviation equal to quarter the size of the domain:

$$\sigma_v = 2.5 \quad \sigma_\omega = 0.5$$
$$\mu_v = 6 \quad \mu_\omega = 1$$

We can thus find the probability $p(v_0, \omega_0)$ of a certain pair v_0, ω_0 being chosen using the bell curve equation. If the two bell curve equations for v_0 and ω_0 are $A(v)$ and $B(\omega)$: $p(v_0, \omega_0) = 0.005A(v_0) \cdot 0.002B(\omega_0)$. This is because the increment on v_0 was 0.005 and the increment on ω_0 was 0.002. Assigning these 'weightings' to each of our v_0, ω_0 pairs, we can find the overall probability P of a coin landing on its side by:

$$\begin{aligned} P &= \frac{\sum_{i=0}^{35} p(v_{0,i}, \omega_{0,i})}{\int_1^{11} A(v)\ dv \cdot \int_0^2 B(\omega)\ d\omega} \\ &= \frac{\sum_{i=0}^{35} p(v_{0,i}, \omega_{0,i})}{0.911069746223} \\ &\approx 0.000011113 \\ &\approx 0.0011113\% \end{aligned}$$

5 Conclusion

So we have achieved a final result that the probability of a coin roughly the shape of a Canadian quarter, in Earth-like conditions, has a probability of landing on its edge of $\sim 0.0011113\%$. Previous studies [1] have been conducted on this subject with a more experimental approach. Their results are in strong agreement with the theoretical results obtained in this simulation - most indicate a probability of around $\frac{1}{60000} \approx 0.0016667\%$. This holds a strong agreement with the results obtained in this study.

It should be noted that there were a few approximations made in this experiment that prevent the current model from being generalized to a cylinder of arbitrary thickness T, since the moment of inertia was approximated to a simpler expression assuming that thickness was small compared to radius R. Furthermore, extreme situations with high initial velocity or high initial angular velocity will show that the model departures from reality, due to effects from friction and air resistance. However, for realistic conditions that closely resembled that on earth, it is clear that our model is closely aligned with reality.

In our analysis of the simulation results, there was a simple assumption made that the distribution of initial velocity and initial angular velocity is shaped in a bell curve. The parameters of the bell curve, the standard deviation σ and average μ were approximated to match real world conditions as much as possible – but it is possible that errors were made in the approximation. This may have skewed results from the raw collected data.

References

[1] Murray, D. and Teare, S. Probability of a tossed coin landing on edge. SAO/NASA ADS Physics, 1993